Herleitung der gemeinsame Posteriori der festen und zufälligen Effekte, $p(\boldsymbol{\beta}, \boldsymbol{b}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{b})p(\boldsymbol{\beta})p(\boldsymbol{b})$, unter der Annahme von Normalverteilungen.

$$
\begin{aligned}
p(\boldsymbol{\beta}, \boldsymbol{b}|\boldsymbol{y}) &\propto p(\boldsymbol{y}|\boldsymbol{\beta}, \boldsymbol{b})p(\boldsymbol{\beta})p(\boldsymbol{b}) \\
&\propto \exp\left(-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{Zb})'\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{Zb}) - \frac{1}{2}(\boldsymbol{\beta} - \boldsymbol{m})'\boldsymbol{M}^{-1}(\boldsymbol{\beta} - \boldsymbol{m}) - \frac{1}{2}\boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b}\right)
\end{aligned}
$$

Betrachte was im Exponenten steht, mit $-\frac{1}{2}$ ausgeklammert:

$$
\begin{aligned}
&= (\boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{Zb})'\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{Zb}) + (\boldsymbol{\beta} - \boldsymbol{m})'\boldsymbol{M}^{-1}(\boldsymbol{\beta} - \boldsymbol{m}) + \boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b} \\
&= \boldsymbol{y}'\boldsymbol{R}^{-1}\boldsymbol{y} + \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X\beta} + \boldsymbol{b}'\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Zb} - 2\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} - 2\boldsymbol{b}'Z'\boldsymbol{R}^{-1}\boldsymbol{y} + 2\boldsymbol{b}'Z'\boldsymbol{R}^{-1}\boldsymbol{X\beta} \\
&\quad + \boldsymbol{\beta}'\boldsymbol{M}^{-1}\boldsymbol{\beta} + \boldsymbol{m}'\boldsymbol{M}^{-1}\boldsymbol{m} - 2\boldsymbol{\beta}\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b} \\
&\propto \boldsymbol{\beta}'(\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} + \boldsymbol{M}^{-1})\boldsymbol{\beta} + \boldsymbol{b}'(\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1})\boldsymbol{b} + \underbrace{2\boldsymbol{b}'(\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X})\boldsymbol{\beta}}_{=\boldsymbol{b}'(\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X})\boldsymbol{\beta} + \boldsymbol{\beta}'(\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z})\boldsymbol{b}} - 2\boldsymbol{\beta}'(\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y}) - 2\boldsymbol{b}'(Z'\boldsymbol{R}^{-1}\boldsymbol{y}) \\
&= \begin{pmatrix}\boldsymbol{\beta}'(\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} + \boldsymbol{M}^{-1}) + \boldsymbol{b}'(Z'\boldsymbol{R}^{-1}\boldsymbol{X}) & \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{b}'(\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1})\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta} \\ \boldsymbol{b}\end{pmatrix} \\
&\quad - 2\begin{pmatrix}\boldsymbol{\beta}' & \boldsymbol{b}'\end{pmatrix}\begin{pmatrix}\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{y}\end{pmatrix} \\
&= \begin{pmatrix}\boldsymbol{\beta}' & \boldsymbol{b}'\end{pmatrix}\begin{pmatrix}\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} + \boldsymbol{M}^{-1} & \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta} \\ \boldsymbol{b}\end{pmatrix} - 2\begin{pmatrix}\boldsymbol{\beta}' & \boldsymbol{b}'\end{pmatrix}\begin{pmatrix}\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{y}\end{pmatrix}
\end{aligned}
$$

Dann erkennt man, dass die Dichte

$$
p(\boldsymbol{\beta}, \boldsymbol{b}|\boldsymbol{y}) \propto \exp\left(-\frac{1}{2}\left(\begin{pmatrix}\boldsymbol{\beta} \\ \boldsymbol{b}\end{pmatrix}'\underbrace{\begin{pmatrix}\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} + \boldsymbol{M}^{-1} & \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1}\end{pmatrix}}_{=\Sigma^{-1}}\begin{pmatrix}\boldsymbol{\beta} \\ \boldsymbol{b}\end{pmatrix} - 2\begin{pmatrix}\boldsymbol{\beta} \\ \boldsymbol{b}\end{pmatrix}'\underbrace{\begin{pmatrix}\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{y}\end{pmatrix}}_{=\Sigma^{-1}\mu}\right)\right)
$$

eine multivariate Normalverteilung mit

$$
\begin{aligned}
\Sigma &= \begin{pmatrix}\boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{X} + \boldsymbol{M}^{-1} & \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} + \boldsymbol{G}^{-1}\end{pmatrix}^{-1} \\
\mu &= \Sigma\begin{pmatrix}\boldsymbol{M}^{-1}\boldsymbol{m} + \boldsymbol{X}'\boldsymbol{R}^{-1}\boldsymbol{y} \\ \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{y}\end{pmatrix}
\end{aligned}
$$

ist.
In der kompakten Notation folgt damit

$$
\begin{aligned}
&\left[\begin{array}{c}\boldsymbol{\beta} \\ \boldsymbol{b}\end{array}\right]\Bigg|\,\boldsymbol{y} \sim N(\boldsymbol{\mu}_{\beta,b}, \boldsymbol{\Sigma}_{\beta,b}) \;\text{ mit} \\
&\quad \boldsymbol{\Sigma}_{\beta,b} = \left(\boldsymbol{C}'\boldsymbol{R}^{-1}\boldsymbol{C} + \boldsymbol{A}\right)^{-1}; \quad \boldsymbol{C} = [\boldsymbol{X}|\boldsymbol{Z}]; \quad \boldsymbol{A} = \left[\begin{array}{cc}\boldsymbol{M}^{-1} & 0 \\ 0 & \boldsymbol{G}^{-1}\end{array}\right] \\
&\quad \boldsymbol{\mu}_{\beta,b} = \boldsymbol{\Sigma}_{\beta,b}\left(\tilde{\boldsymbol{m}} + \boldsymbol{C}'\boldsymbol{R}^{-1}\boldsymbol{y}\right); \quad \tilde{\boldsymbol{m}} = \left[\begin{array}{c}\boldsymbol{M}^{-1}\boldsymbol{m} \\ \boldsymbol{0}\end{array}\right]
\end{aligned}
$$