Gemeinsamen Verteilung von b und y

$$
\begin{aligned}
& \boldsymbol{y}|\boldsymbol{b} \sim N(\boldsymbol{X}\beta + \boldsymbol{Z}\boldsymbol{b}, \boldsymbol{R}), \qquad \boldsymbol{b} \sim N(\boldsymbol{0}, \boldsymbol{G}) \\
\Rightarrow \quad & p(\boldsymbol{y}, \boldsymbol{b}) = p(\boldsymbol{y}|\boldsymbol{b})p(\boldsymbol{b}) \\
= \quad & \frac{1}{(2\pi)^{n/2}|\boldsymbol{R}|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\beta - \boldsymbol{Z}\boldsymbol{b})'\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{X}\beta - \boldsymbol{Z}\boldsymbol{b})\} \cdot \\
& \frac{1}{(2\pi)^{q/2}|\boldsymbol{G}|^{1/2}} \exp\{-\frac{1}{2}\boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b}\} \\
\overset{(1)}{=} \quad & \frac{1}{(2\pi)^{(n+q)/2}\left|\begin{pmatrix} \boldsymbol{V} & \boldsymbol{Z}\boldsymbol{G} \\ \boldsymbol{G}\boldsymbol{Z}' & \boldsymbol{G} \end{pmatrix}\right|^{1/2}} \cdot \\
& \quad \exp\{-\frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\beta - \boldsymbol{Z}\boldsymbol{b})'\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{X}\beta - \boldsymbol{Z}\boldsymbol{b}) - \frac{1}{2}\boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b}\}
\end{aligned}
$$

mit

$$
\left|\begin{pmatrix} \boldsymbol{V} & \boldsymbol{Z}\boldsymbol{G} \\ \boldsymbol{G}\boldsymbol{Z}' & \boldsymbol{G} \end{pmatrix}\right| = |\boldsymbol{G}||\boldsymbol{V} - (\boldsymbol{Z}\boldsymbol{G})\boldsymbol{G}^{-1}(\boldsymbol{G}\boldsymbol{Z}')| = |\boldsymbol{G}||\boldsymbol{R}|. \tag{1}
$$

Es ist

$$\begin{aligned}
&\begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix}' \begin{pmatrix} \boldsymbol{V} & \boldsymbol{Z}\boldsymbol{G} \\ \boldsymbol{G}\boldsymbol{Z}' & \boldsymbol{G} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix} \\
\overset{(2),(3)}{=} &\begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix}' \begin{pmatrix} \boldsymbol{R}^{-1} & -\boldsymbol{R}^{-1}\boldsymbol{Z} \\ -\boldsymbol{Z}'\boldsymbol{R}^{-1} & \boldsymbol{G}^{-1} + \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z} \end{pmatrix} \begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix} \\
= \quad &(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{b})'\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{Z}\boldsymbol{b}) + \boldsymbol{b}'\boldsymbol{G}^{-1}\boldsymbol{b}
\end{aligned}$$

mit der Schur-Komplement-Formel

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}^{-1} = \begin{pmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \\ -\boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & (\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \end{pmatrix} \tag{2}$$

und der Woodbury-Formel

$$(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} = \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1}. \tag{3}$$

Hier:

$$\begin{aligned}
(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} &= (\boldsymbol{V} - \boldsymbol{Z}\boldsymbol{G}\boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{Z}')^{-1} = \boldsymbol{R}^{-1}, \\
(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} &= \boldsymbol{G}^{-1} + \boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z}\boldsymbol{G}\boldsymbol{G}^{-1} = \boldsymbol{G}^{-1} + \boldsymbol{Z}'\boldsymbol{R}^{-1}\boldsymbol{Z}.
\end{aligned}$$

Es folgt:

$$p(\boldsymbol{y}, \boldsymbol{b}) = \frac{1}{(2\pi)^{(n+q)/2} \left| \begin{pmatrix} \boldsymbol{V} & \boldsymbol{ZG} \\ \boldsymbol{GZ}' & \boldsymbol{G} \end{pmatrix} \right|^{1/2}} \cdot \exp\left\{ -\frac{1}{2} \begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\beta \\ \boldsymbol{b} \end{pmatrix}' \begin{pmatrix} \boldsymbol{V} & \boldsymbol{ZG} \\ \boldsymbol{GZ}' & \boldsymbol{G} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{y} - \boldsymbol{X}\beta \\ \boldsymbol{b} \end{pmatrix} \right\}$$

und daher

$$\begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{b} \end{pmatrix} \sim N\left(\begin{pmatrix} \boldsymbol{X}\beta \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{V} & \boldsymbol{ZG} \\ \boldsymbol{GZ}' & \boldsymbol{G} \end{pmatrix} \right).$$